MULTIPLICATIVE STRUCTURES IN EQUIVARIANT HOMOTOPY THEORY

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The primary sources for this talk are [4] and the fourth chapter of [1].

0. Motivation: the Hill-Hopkins-Ravenel Norm

The biggest new element of equivariant homotopy theory introduced by looking at ring spectra is that of *norms*, or multiplicative transfers. Recall from field theory that there are two "averaging maps" for a finite Galois extension: the trace and the norm. One of them is defined as a sum over the action of the Galois group, and the other is defined as a product. The trace is generalized by the additive transfers appearing in Mackey functors. The classical norm from algebraic number theory is generalized by *multiplicative transfers*, which we usually call norms. The most important example of a norm was constructed by Hill-Hopkins-Ravenel in their paper [6], and so I'll begin by describing this construction. We'll then talk about a combinatorial generalization of this, $\mathcal{N}_\infty$-operads and their algebras, and how they describe multiplicative transfers; and finally, we'll look at the formalism of (incomplete) Tambara functors, which play the role for π_0 of equivariant rings that Mackey functors play for π_0 of equivariant spectra.

Definition 0.1. Let G be a finite group. A *G-$\mathbb{E}_\infty$-ring* is an object of $\mathrm{CAlg}(\mathrm{Sp}^G)$.

Construction 0.2 (HHR Norm)**.**

i) The *tensor induction* from H up to G is given by $X \mapsto \bigwedge_{j\in G/H} X_j$.
ii) The *change of universe* functor $I_{U'}^U : \mathrm{Sp}^G[U'] \to \mathrm{Sp}^G[U]$ is given on orthogonal G-spectra by $I_{U'}^U X(V) = J_G(\mathbb{R}^n, V) \wedge_{O(n)} X(\mathbb{R}^n)$, where $\dim V = n$ and $J_G(\mathbb{R}^n, V) = O(V)_+ \wedge_{O(\mathbb{R}^n - V)} \mathbb{S}^{\mathbb{R}^n - V}$ is the Thom spectrum for orthogonal complements. One can show that this is an equivalence of categories; this depends crucially on the structural properties of orthogonal spectra, since genuine equivariant G-spectra are modeled by orthogonal spectra with G-action.
iii) The *norm* functor $N_H^G : \mathrm{Sp}^H \to \mathrm{Sp}^G$ is given by changing to the trivial H-universe, taking the tensor induction up to G, giving the resulting spectrum the canonical action of the wreath product $\Sigma_{|G/H|} \wr H$, and finally changing back to a complete H-universe.

Note that this is defined for both spaces and spectra.

Theorem 0.3. *The norm functor is symmetric monoidal and preserves sifted colimits; and, moreover, the norm behaves correctly with respect to suspensions by representation spheres*[1] *and with respect to taking geometric fixed-points.*

Date: March 26, 2024.

[1]Specifically, it becomes suspension by a representation sphere for the induced representation.

Corollary 0.4. *The norm functor descends to a functor on G-$\mathbb{E}_\infty$-rings, which is left adjoint to restriction.*

Okay, good. Talk done, I guess. But...what is the actual structure of a G-$\mathbb{E}_\infty$-ring? And aren't there examples of "equivariant commutative rings" with only some transfers? To understand this properly, we'll need to describe equivariant rings in terms of operads.

1. Operads

We begin by recalling the definition of an operad.[2]

Definition 1.1. Let $\mathscr{C}$ be a symmetric monoidal category. An *operad* O in $\mathscr{C}$ consists of the following data:

i) For each $n \in \mathbb{N}$, an object $O(n) \in \mathscr{C}$ together with a right Σ_n-action on $O(n)$;
ii) A "unit" map $\mathbb{1}_{\mathscr{C}} \to O(1)$; and
iii) For each k and each $n_1, \ldots, n_k$, composition maps $O(k) \otimes O(n_1) \otimes \cdots \otimes O(n_k) \to O(n_1 + \cdots + n_k)$.

These maps must satisfy composition identities corresponding to grafting of (symmetric rooted) trees; in particular, they must be equivariant with respect to the appropriate symmetric group actions. (Identifying trees which differ by collapse of an internal edge yields associativity and unitality identities.)

Definition 1.2. A map of operads is a levelwise map commuting with the Σ_n-actions, the identity, and the composition maps.

Suppose $\mathscr{D}$ is a symmetric monoidal category enriched over $\mathscr{C}$. Then every object $d \in \mathscr{D}$ has an *endomorphism operad*, $\mathscr{E}nd(d)$, whose n-ary operations are given by $\mathrm{Hom}(d^{\otimes n}, d)$ with the evident permutative action, identity, and composition maps.

Definition 1.3. An *algebra* over an operad O consists of an object d and an a map $O \to \mathscr{E}nd(d)$. (If the category is tensored over $\mathscr{C}$ and has finite colimits, this is the same as a map $O(n) \otimes_{\Sigma_n} X^{\otimes n} \to X$ for each n which satisfy appropriate relations.)

We will generally consider operads in the categories Set, Top, and GTop in this talk; by default, operads will be assumed to be internal to Top, with operads internal to GTop referred to as G-operads. With this convention, we can endow the category of operads with a standard model structure presenting the category of ∞-operads. Much like how ∞-categories can be described using the Joyal model structure on simplicial sets, ∞-operads can be described using the Cisinski-Moerdijk model structure on dendroidal sets ([5]), which replaces simplices with rooted trees.

A particularly important operad in ordinary stable homotopy theory is the $\mathbb{E}_\infty$ operad.

Definition 1.4. An operad O is called $\mathbb{E}_\infty$ if each $O(n)$ is contractible with free Σ_n-action.

[2]For the purpose of this talk, "operad" really means "monochrome symmetric operad". There is no loss of generality in assuming symmetry, because planar operads admit a fully faithful embedding into symmetric ones.

Example 1.5. Fix n, and write $\mathbb{E}_k$ for the operad such that $\mathbb{E}_k(n)$ is the space of disjoint rectilinear embeddings of n k-dimensional cubes into a k-dimensional cube. This is called the little n-cubes operad. We have a natural map $\mathbb{E}_n \to \mathbb{E}_{n+1}$ given by taking the product with an interval, and the colimit as $n \to \infty$ is called $\mathbb{E}_\infty$, the little cubes operad. As the n increases, the connectivity of each operator space of $\mathbb{E}_n$ increases, so the operator spaces of $\mathbb{E}_\infty$ are contractible; that is, it is an $\mathbb{E}_\infty$-operad, as the name suggests.

Proposition 1.6. *All $\mathbb{E}_\infty$-operads are equivalent. (This justifies our reference to "the" $\mathbb{E}_\infty$ operad.)*

Proof. If O and O' are two $\mathbb{E}_\infty$-operads, each of the objects in $O(n) \leftarrow O(n) \times O'(n) \to O'(n)$ is contractible with free Σ_n-action. Consequently, the maps are equivalences. □

2. $\mathcal{N}_\infty$-Operads and Indexing Systems

Commutative algebra and algebraic geometry in the ordinary category of spectra are done over $\mathbb{E}_\infty$-rings, i.e. algebras in Sp over the $\mathbb{E}_\infty$ operad. To describe an appropriate notion of commutative ring spectrum in the equivariant context, we will need to construct an equivariant analogue of $\mathbb{E}_\infty$. Whereas ordinary multiplication is parameterized by tensoring with sets (i.e. the coproduct), equivariant multiplication ought to be parameterized by tensoring with G-sets. Since we want everything to be genuine, and we may not have all multiplicative transfers, this will require us to work with families of subgroups.

Definition 2.1. Let G be a finite group.

i) A *family of subgroups* of G is a subset of the subgroup lattice of G which is closed under conjugation and taking subgroups.
ii) If $\mathscr{F}$ is a family of subgroups of G, the *classifying space* for $\mathscr{F}$ is the G-space $E\mathscr{F}$ such that $E\mathscr{F}(G/H) = *$ if $H \in \mathscr{F}$ and $\varnothing$ otherwise.

This additional structure will give rise to multiplicative transfers between subgroups, or "norms", the first letter of which gives its name to this class of operads.

Definition 2.2. A G-operad O is called $\mathcal{N}_\infty$ if for all n,

i) The Σ_n-action on $O(n)$ is free,
ii) $O(n) = E\mathscr{F}_n$ for some family $\mathscr{F}_n$ of subgroups of $G \times \Sigma_n$ containing $G \times \{1\}$, and
iii) The underlying space of $O(n)$ is contractible.

Example 2.3. Let U be a G-universe, and take $\mathcal{L}_U(n)$ to be the space of linear isometries $U^n \to U$. Then the linear isometries operad $\mathcal{L}_U$ is an $\mathcal{N}_\infty$-operad, since the space of linear isometries between any infinite-dimensional inner product spaces is contractible.

Unlike in the nonequivariant case, however, the linear isometries operad is not the only $\mathcal{N}_\infty$-operad. Not only that, but it isn't even the only $\mathcal{N}_\infty$-operad that can be constructed from this universe!

Example 2.4. Let $V \subset U$ be a subrepresentation, and let R_V be the space of distance-reducing self-embeddings of V with G acting by conjugation. Define a *Steiner path* to be a map $I \to R_V$ with $1 \mapsto \mathrm{id}$, and set $\mathcal{K}(V)(n)$ to be the G-space of n-tuples of Steiner paths f_i such that the self-embeddings $f_i(0)$ have disjoint images. Equip these spaces with the evident operad structure and take the colimit over subrepresentations $V \subset U$ to get $\mathcal{K}(U)$, the *Steiner operad* associated to the universe U.

Theorem 2.5. *If $N \lhd G$ and U_N is the G-universe generated by $\mathbb{R}[G/N]$, $\mathcal{L}(U_N) \simeq \mathcal{K}(U_N)$. However, as long as G has at least three elements, there exists an (incomplete) G-universe U such that $\mathcal{L}(U) \not\simeq \mathcal{K}(U)$.*

This result follows from the classification of $\mathcal{N}_\infty$-operads in terms of combinatorial data called *indexing systems*.

Definition 2.6. Let $\otimes$Cat denote the category of small symmetric monoidal categories and strong monoidal functors. Then we define a *symmetric monoidal categorical coefficient system* (or SMCCS) for G to be a functor $\mathcal{O}_G^{op} \to \otimes\mathrm{Cat}$.

Definition 2.7. Write $\underline{\mathrm{Top}}$ for the SMCCS on G that sends G/H to the category of H-sets. An *indexing system* is a sub-SMBCCS I of $\underline{\mathrm{Top}}$ satisfying the following closure conditions:

i) $I(G/H)$ contains all trivial H-sets
ii) $I(G/H)$ is closed under Cartesian products
iii) $I(G/H)$ is closed under taking subobjects
iv) If $H/K \in I(H)$ and $T \in I(K)$, then $H \times_K T \in I(H)$ (closure under self-induction).

We write $\mathcal{I}$ for the category of indexing systems.[3]

We will show that indexing systems completely describe the homotopy theory of $\mathcal{N}_\infty$-operads.

Definition 2.8. Fix a finite group G and an $\mathcal{N}_\infty$-operad O, and let $H \leq G$. We say an H-set of cardinality n is *admissible* if the graph of the structure morphism $H \to \Sigma_n$ is in the nth family of subgroups $\mathscr{F}_n$ associated to O. If every finite H-set is admissible, we say O is a *complete* $\mathcal{N}_\infty$-operad.

Remark 2.9. It should be noted that every subgroup $\Gamma \in \mathscr{F}_n$ is the graph of a homomorphism. This is because Σ_n is required to act freely on $O(n)$, so any such Γ must have trivial intersection with $\{1\} \times \Sigma_n$ by the definition of $E\mathscr{F}_n$. Consequently, the data of these families of subgroups can equivalently be encoded as the choice of admissible H-sets for each $H \leq G$. We will see that these are the source of multiplicative transfers.

Theorem 2.10. *Define a functor $\mathcal{N}_\infty Op \to \mathcal{I}$ by sending an operad O to the indexing system $\underline{O}$ which associates to G/H the category of admissible H-sets for O. This functor induces an equivalence $h\mathcal{N}_\infty Op \to \mathcal{I}$.*

Proof sketch. In short, we need to show that the conditions defining an $\mathcal{N}_\infty$-operad match up with the conditions defining an indexing system. Suppose we have an $\mathcal{N}_\infty$-operad O, and define a functor $I : \mathcal{O}_G^{op} \to \otimes\mathrm{Cat}$ as described above. Symmetric monoidality amounts to closure under coproducts, which follows by applying the composition map $O(2) \times O(s) \times O(t) \to O(s+t)$, so it remains to check the closure conditions.

[3]This is really just a poset.

The containment of trivial sets in $I(G/H)$ corresponds to the containment of $H \times \{1\}$ in $\mathscr{F}_n$. For the subobject condition, suppose an admissible H-set splits as a coproduct, so the associated morphism $H \to \Sigma_n$ factors through some block subgroup $\Sigma_s \times \Sigma_t$ (where $s+t=n$ and $s,t>0$). Let Γ be the associated subgroup of $G \times \Sigma_n$, which is a subgroup of $G \times \Sigma_s \times \Sigma_t$. Then it is enough to show that the image of Γ under the projection to $G \times \Sigma_s$ is admissible. Using the composition map $O(n) \times O(1)^s \times O(0)^t \to O_s$ reduces us to showing that $(O(n) \times O(1)^s \times O(0)^t)^\Gamma = *$; but the action on the first factor has nonempty fixed points by assumption, and the same follows for $O(1)^s \times O(0)^t$ by restricting along the diagonal and using contractibility of $O(0)$ and $O(1)$.

A similar argument using the wreath product shows the cartesian product condition, and self-induction follows from an explicit computation.

The above shows that $O \mapsto \underline{O}$ is a well-defined fully faithful functor $h\mathcal{N}_\infty Op \to \mathcal{I}$. The essential surjectivity of this functor was proven independently in three different papers in 2017; in particular, Rubin constructed ([7]) an $\mathcal{N}_\infty$-operad for any indexing system in terms of fibrant replacement in a certain model category of discrete G-operads. □

3. Additive and Multiplicative Transfers

Now let's see how admissible sets give rise to transfer maps. Suppose T is an O-admissible H-set of cardinality t (for some $H \leq G$) and write Γ_T for the associated subgroup of $G \times \Sigma_t$. Then $\mathrm{Map}(G \times \Sigma_t/\Gamma_T, O(t)) \cong O(t)^{\Gamma_T} \simeq *$, and for any space X, $(G \times \Sigma_t/\Gamma_T) \times_{\Sigma_t} X^t \cong G \times_H \mathrm{Map}(T, X)$. Thus if X is an O-algebra in GTop, we get a composition

$$G \times_H \mathrm{Map}(T, X) \to (G \times \Sigma_t/\Gamma_T) \times_{\Sigma_t} X^t \to O(t) \times_{\Sigma_t} X^t \to X,$$

where the last map comes from the operadic multiplication. Writing $N^T X := G \times_X \mathrm{Map}(T, X^H)$, any map of admissible H-sets $T \to S$ induces a map $N^T X \to N^S X$ unique up to contractible choice. Taking fixed points and then taking homotopy groups, we get the desired transfers.

Theorem 3.1. *Let O be an $\mathcal{N}_\infty$-operad for a finite group G, $H \leq G$ a subgroup, X an O-algebra in GTop, $k \in \mathbb{N}$, and $T \to S$ a map of admissible H-sets. Then we get a unique, natural abelian group map $\pi_k((N^T X)^H) \to \pi_k((N^S X)^H)$.*

Thus whenever H/K is an admissible H-set, the canonical map $H/K \to H/H$ gives rise to a map $tr_K^H : \pi_k X^K \to \pi_k X^H$. Together with the restriction map, this transfer satisfies a double coset formula.

Theorem 3.2. *Suppose H/K is an admissible H-set, and let K' be a subgroup of H. Then the following formula holds:*

$$res^H_{K'} tr^H_K = \bigoplus_{g \in K' \backslash H/K} tr^{K'}_{K' \cap gKg^{-1}} res^K_{K' \cap gK'g^{-1}}.$$

That is to say, for an admissible H-set T, the transfer associated to the K'-set $T^{K'}$ is obtained by taking the transfer associated to T and restricting down to K'.

So far, we've worked with $\mathcal{N}_\infty$-algebras in the category of G-spaces. If we replace $(G\mathrm{Top}, \times)$ with $(\mathrm{Sp}^G, \wedge)$, however, we get *multiplicative* transfer maps on homotopy groups associated to admissible H-sets. All the results of this section carry over more-or-less unchanged, with the caveat that we must assume the operadic action "interchanges with itself". This means that the structure maps $X^{\wedge n} \to X$ of the O-algebra structure are O-algebra maps. Blumberg-Hill have conjectured ([4]) that the derived tensor product of $\mathcal{N}_\infty$-operads is an $\mathcal{N}_\infty$-operad, which would imply that this is always that case.

It is worth noting that these transfers reproduce the Hill-Hopkins-Ravenel norm ([6]), so this is the systematization we were looking for.

4. Tambara Functors

There are two different multiplicative analogues of Mackey functors, one richer than the other. First, recall the notion of a Green functor.

Definition 4.1. A *Green functor* is a commutative monoid in the category of Mackey functors with the Day convolution product. That is, it is a Mackey functor R such that

i) R(G/H) is a ring
ii) The restriction maps are all ring homomorphisms
iii) The transfer map $R(G/K) \to R(G/H)$ is a homomorphism of $R(G/H)$-modules.

The last condition implies the push-pull relation, of which Theorem 3.2 is one form.

Green functors are all well and good, but this is really the minimal amount of structure we could expect for an equivariant ring. On the other end of this are Tambara functors, which have multiplicative transfers, or "norms", as well as additive transfers.

Definition 4.2. Let $\mathscr{C}$ be a locally Cartesian-closed category. The *category of bispans of* $\mathscr{C}$ is the (weak) $(2,1)$-category $\mathrm{Bispan}(\mathscr{C})$ in which

i) The objects are the objects of $\mathscr{C}$
ii) A morphism from X to Y is a "bispan" $X \leftarrow S \to T \to Y$
iii) A 2-morphism between two bispans is an isomorphism of diagrams which is the identity on X and Y.

Composition of bispans is given by taking pullbacks followed by so-called "exponential diagrams".

We write $\widetilde{\mathrm{Bispan}}(\mathscr{C})$ for the category obtained from $\mathrm{Bispan}(\mathscr{C})$ by group-completing the hom-categories with respect to the coproduct.

Definition 4.3. A *G-Tambara functor* is an additive functor $\widetilde{\mathrm{Bispan}}(\mathcal{O}_G) \to \mathrm{Ab}$.

These are like Mackey functors, but they have an extra transfer map encoded by the extra map in a bispan. (A more detailed overview can be found in [8].) Recall, though, that we don't have every possible transfer; if we're dealing with $\mathcal{N}_\infty$-rings, we'll only have transfers associated to admissible H-sets. This leads to the notion of an "incomplete" Tambara functor.

Definition 4.4. Let $\mathscr{D}$ be a wide, pullback-stable, coproduct-complete subcategory of $G\mathrm{Set}$; we say that $\mathscr{D}$ is an *indexing category* ([2]). Then we write $P^G_{\mathscr{D}}$ for the subcategory[4] of $\widetilde{\mathrm{Bispan}}(G\mathrm{Set})$ of bispans in which the middle morphism in in $\mathscr{D}$. We call an additive functor $P^G_{\mathscr{D}} \to \mathrm{Ab}$ a *$\mathscr{D}$-Tambara functor* ([3]).

Remark 4.5. There is a natural order-preserving map from the poset of indexing categories to the poset of indexing systems, given by sending $\mathscr{D}$ to the system $I_{\mathscr{D}}$ which sends G/H to $\mathscr{D}_{/(G/H)}$. It is not too difficult to show ([3]) that this is an isomorphism.

The results of the previous sections on π_0 can be summed up as follows.

Theorem 4.6. *Let O be an $\mathcal{N}_\infty$-operad, and write $\mathscr{D}$ for the indexing category corresponding to the associated indexing system. Then for any O-algebra X in Sp^G, $\pi_0(X)$ is naturally a $\mathscr{D}$-Tambara functor. If O is a complete $\mathcal{N}_\infty$-operad, $\pi_0(X)$ is a Tambara functor.*

References

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[4] It is not entirely obvious that this is really a category. However, it can be shown that the composition map for bispans respects $\mathscr{D}$ provided that it is pullback-stable.